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CONSTRUCTION OF DISCONTINUOUS SOLUTIONS IN THREE-DIMENSIONAL ELASTICITY THEORY*

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Discontinuous solutions of the equations of elasticity in an elastic space were constructed in /1/ by the generalized integral transform method. Here we adopt a different approach. We first construct a solution for a concentrated jump of displacements and stresses, which is used as Green's function (the influence function), and then construct a solution of the equations of elasticity theory for given displacement and stress jumps distributed over the surface of the defect. In other words, the discontinuous solutions are generated by appropriate integration of the influence functions. The discontinuous solutions are applied to the problem of stress behaviour in the neighbourhood of the vertex of a thin rigid wedge inclusion. Alternative approaches to the construction of discontinuous solutions in the theory of plates and shells are proposed in /2-4/.

1. ***Discontinuous solutions in Cartesian coordinates.*** Consider an elastic space with a defect in the plane $z=0$. This defect is represented by a region Ω such that the stress and displacement fields experience a discontinuity across this region. We introduce the following notation for jumps:

$$\langle u_x\rangle(x, y) = u_x(x, y, -0) - u_x(x, y, +0), \ldots \quad (1.1)$$
$$\langle \sigma_z\rangle(x, y) = \sigma_z(x, y, -0) - \sigma_z(x, y, +0)$$

We assume that the displacement u_x has a concentrated jump at $z=0$ of the form

$$\langle u_x\rangle = [u_x]\,\delta(x)\,\delta(y) \quad (1.2)$$

where $[u_x]$ is the magnitude of the jump and $\delta(x)$ is the delta function.

The equations of equilibrium in displacements in the presence of mass forces have the form /5/

$$\| L \| \{U\} = \mu^{-1}\{V\} \quad (1.3)$$

where $\{U\} = \| u_x u_y u_z \|^T$ is the displacement vector, $\{V\} = \| XYZ \|^T$ is the vector of volume forces, μ is the shear modulus and $\| L \|$ is the 3x3 matrix of differential operators with the components

$$l_{11} = \frac{1}{\varkappa}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad l_{12} = l_{21} = \frac{(1-\varkappa)}{\varkappa}\frac{\partial^2}{\partial x\,\partial y}$$
$$l_{13} = l_{31} = \frac{(1-\varkappa)}{\varkappa}\frac{\partial^2}{\partial x\,\partial z}, \quad l_{22} = \frac{\partial^2}{\partial x^2} + \frac{1}{\varkappa}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
$$l_{23} = l_{32} = \frac{(1-\varkappa)}{\varkappa}\frac{\partial^2}{\partial y\,\partial z}, \quad l_{33} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\varkappa}\frac{\partial^2}{\partial z^2}$$

Here $\varkappa = {}^1/_2(1-2\nu)/(1-\nu)$ and ν is Poisson's ratio.

To solve system (1.3) without mass forces in the presence of a jump (1.2), we apply the integral Fourier transform with respect to z by the generalized scheme of /1/ with the parameter λ. Partitioning the integration interval into subintervals $(-\infty, -0)$ and $(+0, \infty)$, integrating by parts, and conserving the jumps of all functions at $z=0$, we obtain

$$\frac{1}{\varkappa}\frac{\partial^2 u_{x\lambda}}{\partial x^2} + \frac{\partial^2 u_{x\lambda}}{\partial y^2} - \lambda^2 u_{x\lambda} + \frac{(1-\varkappa)}{\varkappa}\frac{\partial^2 u_{y\lambda}}{\partial x\,\partial y} - \frac{(1-\varkappa)\,i\lambda}{\varkappa}\frac{\partial u_{z\lambda}}{\partial x} = f_{1\lambda} \quad (1.4)$$

$$\frac{(1-\varkappa)}{\varkappa}\frac{\partial^2 u_{x\lambda}}{\partial x\,\partial y} + \frac{\partial^2 u_{y\lambda}}{\partial x^2} + \frac{1}{\varkappa}\frac{\partial^2 u_{y\lambda}}{\partial y^2} - \lambda^2 u_{y\lambda} - \frac{(1-\varkappa)\,i\lambda}{\varkappa}\frac{\partial u_{z\lambda}}{\partial y} = f_{2\lambda}$$

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$$-\frac{(1-\varkappa)\,i\lambda}{\varkappa}\left(\frac{\partial u_{x\lambda}}{\partial x}+\frac{\partial u_{y\lambda}}{\partial y}\right)+\frac{\partial^2 u_{z\lambda}}{\partial x^2}+\frac{\partial^2 u_{z\lambda}}{\partial y^2}-\frac{\lambda^2}{\varkappa}u_{z\lambda}=f_{3\lambda}$$

$$f_{1\lambda}=i\lambda\langle u_x\rangle-\left\langle\frac{\partial u_x}{\partial z}\right\rangle-\frac{(1-\varkappa)}{\varkappa}\frac{\partial}{\partial x}\langle u_z\rangle$$

$$f_{2\lambda}=i\lambda\langle u_y\rangle-\left\langle\frac{\partial u_y}{\partial z}\right\rangle-\frac{(1-\varkappa)}{\varkappa}\frac{\partial}{\partial y}\langle u_z\rangle$$

$$f_{3\lambda}=-\frac{(1-\varkappa)}{\varkappa}\left(\frac{\partial}{\partial x}\langle u_x\rangle+\frac{\partial}{\partial y}\langle u_y\rangle\right)-\frac{1}{\varkappa}\left\langle\frac{\partial u_z}{\partial z}\right\rangle+\frac{i\lambda}{\varkappa}\langle u_z\rangle$$

The symbol λ is the Fourier transform parameter of the corresponding function, e.g.,

$$u_{x\lambda}=\int_{-\infty}^{\infty}u_x(x,y,z)\,e^{i\lambda z}\,dz$$

We write Hooke's law for the stresses $\sigma_z,\ \tau_{zx},\ \tau_{zy}$ in the form

$$\sigma_z=\frac{\mu}{\varkappa}\left[\frac{\partial u_z}{\partial z}+(1-2\varkappa)\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)\right] \tag{1.5}$$

$$\tau_{zx}=\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),\quad \tau_{zy}=\mu\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)$$

We stipulate that on crossing the plane $z=0$ only the displacement u_x experiences a jump of the form (1.2), while the displacements $u_y,\ u_z$ and the stresses $\sigma_z,\ \tau_{zx}$ and τ_{zy} are continuous, i.e.,

$$\langle u_y\rangle=\langle u_z\rangle=0,\ \langle\sigma_z\rangle=\langle\tau_{zx}\rangle=\langle\tau_{zy}\rangle=0 \tag{1.6}$$

Now rewriting (1.5) in jumps and using (1.6), we obtain

$$\left\langle\frac{\partial u_x}{\partial z}\right\rangle=\left\langle\frac{\partial u_y}{\partial z}\right\rangle=0,\quad \left\langle\frac{\partial u_z}{\partial z}\right\rangle=-(1-2\varkappa)[u_x]\,\delta'(x)\,\delta(y)$$

Then the right-hand sides of Eqs.(1.4) take the form

$$f_{1\lambda}=i\lambda\,[u_x]\,\delta(x)\,\delta(y),\ f_{2\lambda}=0,\ f_{3\lambda}=-[u_x]\,\delta'(x)\,\delta(y) \tag{1.7}$$

If the mass forces in (1.3) are taken in the form

$$X=-\mu\,[u_x]\,\delta(x)\,\delta(y)\,\delta'(z),\ Y=0,\ Z=-\mu\,[u_x]\delta'(x)\,\delta(y)\,\delta(z)$$

and are Fourier-transformed with respect to z, we obtain (1.7). Therefore, the jump (1.2) can be obtained if two directed concentrated moments /6/ of intensity $M_{xz}=M_{zx}=-\mu\,[u_x]$ are applied at the origin.

We can similarly treat concentrated jumps of the displacements u_y and u_z along z for $z=0$. For instance, the jump $\langle u_y\rangle=[u_y]\,\delta(x)\,\delta(y)$ can be obtained if we take two concentrated moments of intensity $M_{yz}=M_{zy}=-\mu\,[u_y]$. Finally, to obtain the jump $\langle u_z\rangle=[u_z]\,\delta(x)\,\delta(y)$, we should use three dipoles of intensity $D_x=D_y=-\mu\varkappa^{-1}(1-2\varkappa)\,[u_z]$, $D_z=-\mu\varkappa^{-1}\,[u_z]$. Now solving system (1.4) or using the available solutions (from /6/, say), we obtain expressions for the displacements. For brevity we will use matrix notation. We introduce the jump vector

$$\{S_u\}=\|\,[u_x]\,[u_y]\,[u_z]\,\|^T$$

The dependence of the displacement vector on the jump vector is written in the form

$$\{U\}=\|G\|\,\{S_u\} \tag{1.8}$$

where $\|G\|$ is a 3x3 functional matrix with elements $g_{ij}\equiv g_{ij}(x,\ y,\ z)$:

$$g_{11}=-zg_x^{+},\ g_{12}=g_{21}=-3\,(4\pi)^{-1}\,(1-\varkappa)\,xyzr^{-5} \tag{1.9}$$

$$g_{13}=xg_z^{-},\ g_{22}=-zg_y^{+},\ g_{23}=yg_z^{-}$$

$$g_{31}=-xg_z^{+},\ g_{32}=-yg_z^{+},\ g_{33}=-zg_z^{+}$$

$$(g_u^{\pm}=(4\pi)^{-1}r^{-3}\,[\varkappa\pm3\,(1-\varkappa)\,u^2r^{-2}],\ u=x,\ y,\ z;\ r=(x^2+y^2+z^2)^{1/2})$$

The stresses $\tau_{zx},\ \tau_{zy}$ and σ_z may also experience jumps across the plane $z=0$. We can similarly derive the following result. If the stresses τ_{zx} have a concentrated jump $\langle\tau_{zx}\rangle=[\tau_{zx}]\,\delta(x)\,\delta(y)$ at $z=0$, then the solution is identical to Kelvin's solution, with a concentrated force $X = [\tau_{zx}]$ applied at the origin. To obtain a solution for the concentrated jump $\langle\tau_{zy}\rangle = [\tau_{zy}]\,\delta(x)\,\delta(y)$, we need to apply a concentrated force $Y = [\tau_{zy}]$ at the origin. Finally, if $\langle\sigma_z\rangle = [\sigma_z]\,\delta(x)\,\delta(y)$, we need to take $Z = [\sigma_z]$. A dependence similar to (1.8) has the form

$$\{U\} = \|\Gamma\|\,\{S_\sigma\},\ \{S_\sigma\} = \|\,[\tau_{zx}]\,[\tau_{zy}]\,[\sigma_z]\,\|^T \tag{1.10}$$

Using Kelvin's solution /5, 6/, we obtain the elements of the matrix $\|\Gamma\|$

$$\gamma_{11} = \gamma_x,\ \gamma_{12} = \gamma_{21} = (8\pi\mu)^{-1}\,xyr^{-3},\ \gamma_{13} = \gamma_{31} = (8\pi\mu)^{-1}\,(1-\varkappa)\,xzr^{-3} \tag{1.11}$$
$$\gamma_{22} = \gamma_y,\ \gamma_{23} = \gamma_{32} = x^{-1}y\gamma_{13},\ \gamma_{33} = \gamma_z$$
$$(\gamma_u = (8\pi\mu)^{-1}r^{-1}\,[(1+\varkappa) + (1-\varkappa)\,u^2r^{-2}],\ u = x,\ y,\ z)$$

Applying Hooke's law and relationships (1.8) and (1.10), we can find the stresses as a function of the displacement and stress jump vectors:

$$\{\sigma\} = \|T\|\,\{S_u\},\ \{\sigma\} = \|Q\|\,\{S_\sigma\} \tag{1.12}$$

where $\{\sigma\} = \|\,\sigma_x\sigma_y\sigma_z\tau_{xy}\tau_{yz}\tau_{zx}\,\|^T$. The elements of the matrices $\|T\|$ and $\|Q\|$ have the form

$$t_{11} = -3\,(2\pi)^{-1}\,(1-\varkappa)\,\mu xzr^{-5}\,(1-5x^2r^{-2}) \tag{1.13}$$
$$t_{21} = -3\,(2\pi)^{-1}\mu xzr^{-5}\,[(1+\varkappa) - 5\,(1-\varkappa)\,y^2r^{-2}]$$
$$t_{31} = t_{63} = -3\,(2\pi)^{-1}\,(1-\varkappa)\,\mu xzr^{-5}\,(1-5z^2r^{-2})$$
$$t_{41} = -3\,(4\pi)^{-1}\,\mu yzr^{-5}\,[(1-2\varkappa) - 10\,(1-\varkappa)\,x^2r^{-2}]$$
$$t_{61} = (4\pi)^{-1}\mu r^{-3}\,[(4\varkappa-3) + 3\,(1-2\varkappa)\,y^2r^{-2} + 30\,(1-\varkappa)\,x^2z^2r^{-4}]$$
$$t_{12} = -3\,(2\pi)^{-1}\mu yzr^{-5}\,[\varkappa - 5\,(1-\varkappa)\,x^2r^{-2}]$$
$$t_{32} = t_{53} = x^{-1}yt_{31},\ t_{62} = t_{51}$$
$$t_{13} = -(2\pi)^{-1}\mu r^{-3}\,[1 - 3\varkappa y^2r^{-2} - 15\,(1-\varkappa)\,x^2z^2r^{-4}]$$
$$t_{33} = -(2\pi)^{-1}\,(1-\varkappa)\,\mu r^{-3}\,(1 + 6z^2r^{-2} - 15z^4r^{-4})$$
$$q_{11} = -xg_x^+,\ q_{21} = xg_y^-,\ q_{31} = xg_z^-,\ q_{41} = -yg_x^+$$
$$q_{51} = q_{43} = q_{62} = -3\,(4\pi)^{-1}\,(1-\varkappa)\,xyzr^{-5},\ q_{61} = -zg_x^+$$
$$q_{12} = yg_x^-,\ q_{22} = -yg_y^+,\ q_{32} = yg_z^-,\ q_{42} = -xg_y^+,\ q_{52} = -zg_y^+$$
$$q_{13} = zg_x^-,\ q_{23} = zg_y^-,\ q_{33} = -zg_z^+,\ q_{53} = -yg_z^+,\ q_{63} = -xg_z^+$$

The coefficients t_{22}, t_{42}, t_{52} and t_{23} can be obtained respectively from the coefficients t_{11}, t_{41}, t_{61} and t_{13} by making the substitution $(x \rightleftarrows y)$, and t_{51} and t_{43} can be obtained from t_{41} and t_{12} by the substitution $(x \rightleftarrows z)$.

If the concentrated displacement and stress jumps are specified at an arbitrary point (ξ, η), then in expressions (1.9), (1.11) and (1.13) we need to substitute $x - \xi$ for x and $y - \eta$ for y. For example, suppose that the displacement jump u_x is given at the point $x = \xi$, $y = \eta$. Using (1.8), we may write, for instance, $u_x = g_{11}(x-\xi,\ y-\eta,\ z)\,[u_x]$. If the jump $[u_x]$ varies in some region Ω with density $[u_x] = \langle u_x\rangle(x, y)$, then

$$u_x = \iint g_{11}(x-\xi,\ y-\eta,\ z)\,\langle u_x\rangle(\xi,\ \eta)\,d\Omega$$

Here and henceforth, integration is over the region Ω.

We can proceed similarly when there is a defect of an arbitrary nature in the plane $z = 0$ (for instance, a peeling rigid inclusion). In general, we observe jumps in the displacements u_x, u_y, u_z and in the stresses $\tau_{zx}, \tau_{zy}, \sigma_z$ across the defect. Applying relationships (1.8), (1.10) and (1.12), we can write a discontinuous solution containing the given jumps (1.1);

$$\{U^\circ\} = \|K_{11}^{(z)}\|\,\{S_u\} + \|K_{12}^{(z)}\|\,\{S_\sigma\} \tag{1.14}$$
$$\{\sigma^\circ\} = \|K_{21}^{(z)}\|\,\{S_u\} + \|K_{22}^{(z)}\|\,\{S_\sigma\}$$

Here

$$\{U^\circ\} = \|\,u_x^\circ\ u_y^\circ\ u_z^\circ\,\|^T,\ \{\sigma^\circ\} = \|\,\sigma_z^\circ\tau_{yz}^\circ\tau_{zx}^\circ\,\|^T \tag{1.15}$$
$$\{S_u\} = \|\,\langle u_x\rangle\ \langle u_y\rangle\ \langle u_z\rangle\,\|^T,\ \{S_\sigma\} = \|\,\langle\tau_{zx}\rangle\ \langle\tau_{zy}\rangle\ \langle\sigma_z\rangle\,\|^T$$
$$\|K_{11}^{(z)}\| = \begin{Vmatrix} G_{11}^{(z)} & G_{12}^{(z)} & G_{13}^{(z)} \\ G_{21}^{(z)} & G_{22}^{(z)} & G_{23}^{(z)} \\ G_{31}^{(z)} & G_{32}^{(z)} & G_{33}^{(z)} \end{Vmatrix},\quad \|K_{12}^{(z)}\| = \begin{Vmatrix} \Gamma_{11}^{(z)} & \Gamma_{12}^{(z)} & \Gamma_{13}^{(z)} \\ \Gamma_{21}^{(z)} & \Gamma_{22}^{(z)} & \Gamma_{23}^{(z)} \\ \Gamma_{31}^{(z)} & \Gamma_{32}^{(z)} & \Gamma_{33}^{(z)} \end{Vmatrix}$$

$$\| K_{21}^{(z)} \| = \begin{Vmatrix} T_{31}^{(z)} & T_{32}^{(z)} & T_{33}^{(z)} \\ T_{51}^{(z)} & T_{52}^{(z)} & T_{53}^{(z)} \\ T_{61}^{(z)} & T_{62}^{(z)} & T_{63}^{(z)} \end{Vmatrix}, \quad \| K_{22}^{(z)} \| = \begin{Vmatrix} Q_{31}^{(z)} & Q_{32}^{(z)} & Q_{33}^{(z)} \\ Q_{51}^{(z)} & Q_{52}^{(z)} & Q_{53}^{(z)} \\ Q_{61}^{(z)} & Q_{62}^{(z)} & Q_{63}^{(z)} \end{Vmatrix}$$

$$G_{ij}^{(z)}f = \iint g_{ij}(x-\xi,\ y-\eta,\ z) f(\xi,\ \eta)\, d\Omega,$$

$$\Gamma_{ij}^{(z)}f = \iint \gamma_{ij}(x-\xi,\ y-\eta,\ z) f(\xi,\ \eta)\, d\Omega$$

$$T_{ij}^{(z)}f = \iint t_{ij}(x-\xi,\ y-\eta,\ z) f(\xi,\ \eta)\, d\Omega,\ Q_{ij}^{(z)}f = \iint q_{ij}(x-\xi,\ y-\eta,\ z) f(\xi,\ \eta)\, d\Omega$$

where the functions g_{ij}, γ_{ij}, t_{ij} and q_{ij} are given by expressions (1.9), (1.11) and (1.13).

2. ***Discontinuous solutions in cylindrical coordinates.*** In some three-dimensional problems with stress concentration near defects, we need to obtain discontinuous solutions in cylindrical coordinates. The results of Sect.1 may be extended to the case when the displacements and stresses experience jumps near the axes of the cylindrical system of coordinates. Consider jumps along the z' axis of the local coordinate system (x', y', z') shown in Fig.1. Using the results of Sect.1, we can write

$$\{U'\} = \| G' \| \{S_{u'}\} \tag{2.1}$$

where $\{U'\} = \| u_{x'} u_{y'} u_{z'} \|^T$ is the displacement vector in the coordinates (x', y', z'); $\{S_{u'}\} = \| [u_{x'}]\ [u_{y'}]\ [u_{z'}] \|^T$ is the displacement jump vector. The elements of the matrix $\| G' \|$ are given by formulas (1.11), where x, y, z must be replaced with x', y', z', respectively. Denote the displacements in the cylindrical system of coordinates by u_r, u_θ, u_z (Fig.1) and the corresponding jumps concentrated at the point $(\rho,\ \eta)$ by $[u_\rho]$, $[u_\eta]$, $[u_{z'}]$. We have the obvious equalities

$$[u_{x'}] = [u_\rho],\ [u_{y'}] = [u_\eta],\ [u_{z'}] = [u_z]$$

or

$$\{S_{u'}\} = \{S_u\} = \| \ [u_\rho]\ [u_\eta]\ [u_z] \|^T$$

Consider an arbitrary point $M(x', y', z')$. The components of the displacement vector are $u_{x'}$, $u_{y'}$, $u_{z'}$ in coordinates (x', y', z') and u_r, u_θ, u_z in cylindrical coordinates (r, θ, z). The relationship between the displacement components in the two coordinate systems has the form /7/

$$\{U'\} = \| K \| \{U\},\ \{U\} = \| u_r\ u_\theta\ u_z \|^T \tag{2.2}$$

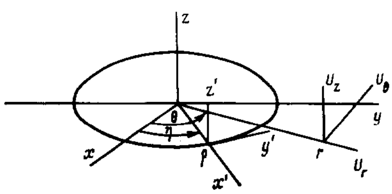

Fig.1

The elements of the matrix $\| K \|$ are the cosines of the angles between the corresponding axes:

$$k_{11} = k_{22} = \cos\gamma,\ k_{12} = -k_{21} = -\sin\gamma,\ k_{33} = 1$$
$$k_{13} = k_{31} = k_{32} = k_{23} = 0,\ \gamma = \theta - \eta$$

Substituting (2.2) in (2.1) and using the orthogonality of the matrix $\| K \|$, we obtain

$$\{U\} = \| G \| \{S_u\},\ \ \| G \| = \| K \|^T \{G'\} \tag{2.3}$$

Multiplying the matrices in (2.3), we obtain all the elements of the matrix $\| G \|$. Then expressing x', y', z' in terms of r, ρ, θ, η, z by the relations

$$x' = r\cos\gamma - \rho,\ \ y' = r\sin\gamma,\ \ z' = z$$

we obtain $(g_{ik} \equiv g_{ik}(r,\ \rho,\ \gamma,\ z))$

$$g_{11} = (4\pi)^{-1}\varkappa z R^{-3}(1-\cos\gamma) - z g^-_{\omega^+\omega^-},\ \ g_{21} = z\sin\gamma g^-_{\rho\omega^+} \tag{2.4}$$
$$g_{31} = -\omega^+ g_{zz}^+,\ \ g_{12} = -z\sin\gamma g^-_{r\omega^-},\ \ g_{13} = -\omega^- g_{zz}^-$$
$$g_{22} = -(4\pi)^{-1} z R^{-3}[\varkappa\cos\gamma + 3(1-\varkappa)\, r\rho R^{-2}\sin^2\gamma]$$
$$g_{32} = -r\sin\gamma g_{zz}^+,\ \ g_{23} = \rho\sin\gamma g_{zz}^-,\ \ g_{33} = -z g_{zz}^+$$
$$(g_{uv}^\pm = (4\pi)^{-1}R^{-3}[\varkappa \pm 3(1-\varkappa)\, uvR^{-2}];\ u, v = z, r, \rho, \omega^+, \omega^-;$$
$$\omega^+ = r\cos\gamma - \rho,\ \ \omega^- = \rho\cos\gamma - r,\ \ R = (r^2 + \rho^2 - 2r\rho\cos\gamma)^{1/2})$$

We can similarly consider the stress jumps in the cylindrical coordinate system. The analogue of (2.3) is

$$\{U\} = \| \Gamma \| \{S_\sigma\} \tag{2.5}$$

where $\{S_\sigma\} = \| [\tau_{z\rho}][\tau_{z\eta}][\sigma_z] \|^T$ is the vector of stress jumps along z at the point (ρ, η). The elements of the matrix $\| \Gamma \|$ are given by

$$\gamma_{11} = (8\pi\mu R)^{-1} [(1+\varkappa)\cos\gamma - (1-\varkappa)\,\omega^+\omega^- R^{-2}] \tag{2.6}$$
$$\gamma_{21} = -\gamma^-_{\rho\omega^+}\sin\gamma, \quad \gamma_{31} = (8\pi\mu)^{-1}(1-\varkappa)\, z\omega^+ R^{-3}$$
$$\gamma_{12} = \gamma^-_{r\omega^-}\sin\gamma,\ \gamma_{22} = (8\pi\mu R)^{-1}[(1+\varkappa)\cos\gamma + (1-\varkappa)\, r\rho R^{-2}\sin^2\gamma]$$
$$\gamma_{32} = (8\pi\mu)^{-1}(1-\varkappa)\, zrR^{-3}\sin\gamma,\ \gamma_{13} = -(8\pi\mu)^{-1}(1-\varkappa)\, z\omega^- R^{-3}$$
$$\gamma_{23} = r^{-1}\rho\gamma_{32},\quad \gamma_{33} = \gamma_{zz}^+$$
$$(\gamma_{uv}^{\pm} = (8\pi\mu R)^{-1}[(1+\varkappa) \pm (1-\varkappa)\, uvR^{-2}];\ u, v = z, r, \rho, \ \omega^+, \omega^-)$$

Applying Hooke's law and expressions (2.3) and (2.5), we can express the stresses in terms of the displacement and stress jump vectors:

$$\{\sigma\} = \| T \| \{S_u\},\ \{\sigma\} = \| Q \| \{S_\sigma\}\ \ (\{\sigma\} = \| \sigma_r\sigma_\theta\sigma_z\tau_{r\theta}\tau_{\theta z}\tau_{zr} \|^T) \tag{2.7}$$

We will write out only those elements of the matrices $\| T \|$ and $\| Q \|$ which are needed later:

$$t_{31} = -3\,(2\pi)^{-1}(1-\varkappa)\,\mu z\omega^+ R^{-5}\,(1 - 5z^2R^{-2}),\ t_{32} = (\omega^+)^{-1}\, r\sin\gamma\, t_{31} \tag{2.8}$$
$$t_{33} = -(2\pi)^{-1}(1-\varkappa)\,\mu R^{-3}\,(1 + 6z^2R^{-2} - 15z^4R^{-4})$$
$$t_{51} = -(4\pi)^{-1}\mu R^{-3}\sin\gamma\,[(4\varkappa - 3) - 3\,(1-2\varkappa)\, r\omega^- R^{-2} - 30\,(1-\varkappa)\,\rho z^2\omega^+ R^{-4}]$$
$$t_{52} = (4\pi)^{-1}\mu R^{-3}\,[(4\varkappa - 3)\cos\gamma - 3\,(1-2\varkappa)\,\omega^+\omega^- R^{-2} + 30\,(1-\varkappa)\, r\rho z^2 R^{-4}\sin^2\gamma]$$
$$t_{53} = (\omega^+)^{-1}\rho\sin\gamma\, t_{31},\ \ t_{63} = -\omega^-\,(\omega^+)^{-1} t_{31}$$
$$t_{61} = (4\pi)^{-1}\mu R^{-3}\,[(4\varkappa - 3)\cos\gamma + 3\,(1-2\varkappa)\, r\rho R^{-2}\sin^2\gamma - 30\,(1-\varkappa)\, z^2\omega^+\omega^- R^{-4}]$$
$$t_{62} = (4\pi)^{-1}\mu R^{-3}\sin\gamma\,[(4\varkappa - 3) - 3\,(1-2\varkappa)\,\rho\omega^+ R^{-2} - 30\,(1-\varkappa)\, rz^2\omega^- R^{-4}]$$
$$q_{31} = \omega^+ g_{zz}^-,\ q_{32} = rg_{zz}^-\sin\gamma,\ q_{33} = -zg_{zz}^+$$
$$q_{51} = zg^-_{\rho\omega^+}\sin\gamma,\ \ q_{52} = -(4\pi)^{-1}zR^{-3}\,[\varkappa\cos\gamma + 3\,(1-\varkappa)\, r\rho R^{-2}\sin^2\gamma]$$
$$q_{53} = -\rho g_{zz}^+\sin\gamma,\ \ q_{61} = (4\pi)^{-1}zR^{-3}\,[-\varkappa\cos\gamma + 3\,(1-\varkappa)\,\omega^+\omega^- R^{-2}]$$
$$q_{62} = -zg^-_{r\omega^-}\sin\gamma,\ \ q_{63} = \omega^- g_{zz}^+$$

The discontinuous solution in cylindrical coordinates may be written in the form (1.14), where

$$\{U^\circ\} = \| u_r^\circ u_\theta^\circ u_z^\circ \|^T,\ \ \{\sigma^\circ\}, = \| \sigma_z^\circ\tau_{zr}^\circ\tau_{z\theta}^\circ \|^T$$
$$\{S_u\} = \| \langle u_\rho\rangle\langle u_\eta\rangle\langle u_z\rangle \|^T,\ \ \{S_\sigma\} = \| \langle\tau_{z\rho}\rangle\langle\tau_{z\eta}\rangle\langle\sigma_z\rangle \|^T$$

and the integral operators in (1.14) act by the rule

$$G_{ij}^{(z)}f = \iint g_{ij}f\,d\Omega,\ \ \Gamma_{ij}^{(z)}f = \iint \gamma_{ij}f\,d\Omega \tag{2.9}$$
$$T_{ij}^{(z)}f = \iint t_{ij}f\,d\Omega,\ \ Q_{ij}^{(z)}f = \iint q_{ij}f\,d\Omega$$

where integration is over the region Ω and the functions $g_{ij}, \gamma_{ij}, t_{ij}, q_{ij}$ depend on r, ρ, γ, z and are given by (2.4), (2.6) and (2.8).

3. ***Application to the problem of stress concentration near a defect.*** The discontinuous solutions (1.14) enable us to obtain a system of integral equations for the unknown jumps by using the conditions on the defect. We will demonstrate this technique for the problem of a rigid thin wedge inclusion peeling from the substrate in the plane $z = 0$; the wedge occupies the region $|\theta| \leqslant \alpha, 0 \leqslant r < \infty$. The stress state of the elastic space is represented as the sum of the basic stress state induced by an external load and the perturbed stress state induced by the defect. Thus,

$$\{U\} = \{U^*\} + \{U^\circ\},\ \{\sigma\} = \{\sigma^*\} + \{\sigma^\circ\} \tag{3.1}$$

where the asterisk identifies the main state variables.

Assume that the lower lip of the inclusion $(z = -0)$ adheres to the elastic space, while the upper lip $(z = +0)$ has separated. Then we have the following conditions on the defect:

$$\sigma_z(r, \theta, +0) = \tau_{zr}(r, \theta, +0) = \tau_{z\theta}(r, \theta, +0) = 0 \tag{3.2}$$
$$u_r(r, \theta, -0) = u_r' - r\omega_z \sin\theta + z\omega_y, \quad u_\theta(r, \theta, -0) = u_\theta' - z\omega_x + r\omega_z \cos\theta, \quad u_z(r, \theta, -0) = u_z' - r\omega_y \cos\theta + r\omega_x \sin\theta$$

where $u_r', u_\theta', u_z', \omega_x, \omega_y, \omega_z$ are respectively the displacements and the angles of rotation of the inclusion as a rigid whole.

Applying (3.1) to realize conditions (3.2) on the defect, we obtain a system of integral equations for the unknown jumps. We must define the kernels of the integral operators in (3.1) for $z = \pm 0$. To this end, we use the relationship

$$\lim z(z^2 + r^2 + \rho^2 - 2r\rho\cos\gamma)^{-3/2} = \pm 2\pi\rho^{-1}\delta(r-\rho)\,\delta(\gamma) \quad (z \to \pm 0)$$

and its analogues obtained by differentiation with respect to r, ρ, γ.

The method of /8, 9/ can be used to analyse the behaviour of jumps in the neighbourhood of a thin rigid wedge inclusion. Without considering the general case, we will analyse in detail the special case when the stresses τ_{zr} and $\tau_{z\theta}$ experience jumps, while the stresses σ_z and the displacements are continuous at $z = 0$. This corresponds to the case when the inclusion is without bending rigidity and adheres to the elastic space at $z = \pm 0$. In this case, the system of integral equations has the form

$$\Gamma_{i1}^{(\circ)}\langle\tau_{z\rho}\rangle + \Gamma_{i2}^{(\circ)}\langle\tau_{z\eta}\rangle = f_i, \quad i = 1,2 \tag{3.3}$$

The operators $\Gamma_{ij}^{(\circ)}$ are evaluated from (2.9) for $z = 0$; f_i are functions which are not given here.

To study the behaviour of the jumps $\langle\tau_{zr}\rangle$ and $\langle\tau_{z\theta}\rangle$ as $r \to 0$, following /8, 9/, we apply the Mellin integral transform with respect to r to the system (3.3). As a result, we obtain the system

$$\int_{-\alpha}^{\alpha} \| K^{(\circ)} \| \{\phi\}\, d\eta = \{F\}$$

$$\{\phi\} = \|\varphi_{1s}\varphi_{2s}\|^T, \quad \varphi_{1s}(\eta) = \int_0^\infty \langle\tau_{z\rho}\rangle \rho^s\, d\rho, \quad \varphi_{2s}(\eta) = \int_0^\infty \langle\tau_{z\eta}\rangle \rho^s\, d\rho$$

$$\{F\} = \|f_{1s}f_{2s}\|^T, \quad f_{is}(\theta) = 8\pi\mu \int_0^\infty f_i(r,\theta) r^{s-1}\, dr, \quad i = 1, 2$$

The elements of the matrix $\|K^\circ\|$ have the form

$$k_{11}^\circ(\gamma) = -s^*\{[s(1-\varkappa) - (1+\varkappa)]\, xP_s(x) + (1-\varkappa)(1+s)P_{s+1}(x)\}$$
$$k_{12}^\circ(\gamma) = s^*[s(1-\varkappa) - (1+\varkappa)]\sin\gamma P_s(x), \quad k_{21}^\circ(\gamma) = s^*[2 + s(1-\varkappa)]\sin\gamma P_s(x)$$
$$k_{22}^\circ(\gamma) = s^*\{[(1+\varkappa) + (1-\varkappa)(1+s)]\, xP_s(x) + (1-\varkappa)(1+s)\, P_{s+1}(x)\}$$
$$(s^* = \pi/\sin\pi s, \quad x = -\cos\gamma)$$

Here $P_s(x)$ is the adjoint Legendre function on the cut /10/.

Using the relationship between Legendre functions and the hypergeometric function, we can show by analytical continuation of the hypergeometric function /10/ that $k_{12}^\circ(\gamma)$ and $k_{21}^\circ(\gamma)$ are continuous functions and

$$k_{11}^\circ(\gamma) = -4\ln(|\gamma|/\alpha) + k_{11}^*(\gamma), \quad k_{22}^\circ(\gamma) = -2(1+\varkappa)\ln(|\gamma|/\alpha) + k_{22}^*(\gamma)$$

where $k_{11}^*(\gamma)$ and $k_{22}^*(\gamma)$ are continuous functions.

We will construct the solution of the system by the method of orthogonal polynomials /1/. To this end, we reduce the system to the iterval $(-1,1)$ and represent the functions $\varphi_{is}(\alpha\theta)$ $(i = 1,2)$ in the form

$$\varphi_{1s}(\alpha\theta) = (1-\theta^2)^{-1/2}\sum_{m=1}^{\infty} X_m T_{m-1}(\theta), \quad \varphi_{2s}(\alpha\theta) = (1-\theta^2)^{-1/2}\sum_{m=1}^{\infty} Y_m T_{m-1}(\theta)$$

The continuous functions $k_{11}^*, k_{12}^\circ, k_{21}^\circ, k_{22}^*$ are approximated by segments of Fourier series in Chebyshev polynomials of the first kind,

$$k_{\lambda\beta}^{*(\circ)}(\alpha\gamma) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{(\lambda\beta)} T_{i-1}(\theta)\, T_{j-1}(\eta), \quad \lambda, \beta = 1, 2$$

The ceofficients $a_{ij}^{(\lambda\beta)}$ are evaluated from standard formulas /11/, for instance

$$a_{ij}^{(11)} = \frac{4}{N^2(1+\delta_{1i})(1+\delta_{1j})}\sum_{p=1}^{N}\sum_{q=1}^{N} k_{11}^*[\alpha(\lambda_p - \lambda_q)]\, T_{i-1}(\lambda_i)\, T_{j-1}(\lambda_q)$$

$$(\lambda_n = \cos[{}^1/_2(2n-1)\pi / N])$$

The orthogonal polynomial method /1/ produces the following coupled system of algebraic equations for the coefficients X_m, Y_m:

$$8\mu_{n-1}\pi^{-1}X_n + \sum_{m=1}^{N}\varepsilon_m a_{nm}^{(11)}X_m + \sum_{m=1}^{N}\varepsilon_m a_{nm}^{(12)}Y_m = f_{1n} \tag{3.4}$$

$$\sum_{m=1}^{N}\varepsilon_m a_{nm}^{(21)}X_m + 4(1+\varkappa)\pi^{-1}\mu_{n-1}Y_n + \sum_{m=1}^{N}\varepsilon_m a_{nm}^{(22)}Y_m = f_{2n}$$

$$(n = 1, 2, \ldots; \mu_0 = \ln 2, \mu_n = n^{-1}, n = 1, 2, \ldots; \varepsilon_1 = 2, \varepsilon_m = 1, m = 2, 3, \ldots)$$

Following /9/, we equate to zero the determinant of system (3.4) and obtain an equation for the exponent s. Below we give the values of s for $\nu = 0.3$ and various α:

$\alpha\pi^{-1}\cdot 10^3$	125	250	375	500	625	750	875
$s\cdot 10^3$	180	255	355	500	588	725	912

To obtain s to three decimal places for $\alpha \leqslant 0.625\pi$, it suffices to take $N = 6$ in system (3.4). As α increases, the convergence becomes poorer and for $\alpha = 0.875\pi$ we must take $N = 9$.

Thus, the jumps $\langle\tau_{zr}\rangle$ and $\langle\tau_{z\theta}\rangle$ behave as $O(r^{s-1})$ as $r \to 0$. The stresses τ_{zr} and $\tau_{z\theta}$ have the same feature.

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